On the decompositions of a reciprocal into the sum and the difference of two reciprocals

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Abstract: This paper deals with the relatively simple problem of determining the decomposition of a reciprocal into the sum or difference of two reciprocals, and of establishing how many such decompositions exist. First, we demonstrate the method on four illustrative cases. We then show that, for a positive integer n, the number of decompositions of $1/n$ into the sum or difference of two reciprocals is determined by the number of divisors of n^2. Afterwards, we illustrate the process of finding decompositions for a specific positive integer. Next, we present the results for an arbitrary positive integer and derive the corresponding explicit formulas. Finally, we present a program created in the computer algebra system Maple 2025 for determining the decompositions for any positive integer, and verify the theoretical results for the previously considered example.

Key-Words: reciprocal, Egyptian fraction, greedy algorithm, Erdős–Straus conjecture, prime factorization, Simon's Favorite Factoring Trick, Diophantine equation, divisor function, computer algebra system Maple 2025

"Every equation is a doorway: some open to truth, others to beauty, all to wonder."
— Anonymous Mathematician (21st century)

1 Introduction

The writing of this paper was inspired by the question: "*How can one determine all possible ways of writing 1/11 as the sum of the reciprocals of two positive integers?*" (see [1]) on Quora – an online forum for sharing knowledge and for better understanding the world. It wa stated that there are two solutions, namely of the forms

$$\frac{1}{11}=\frac{1}{22}+\frac{1}{22} \qquad \text{and} \qquad \frac{1}{11}=\frac{1}{12}+\frac{1}{132}.$$

This paper describes a relatively simple method for determining a decomposition of $1/n$, where n is any positive integer, into the sum of two fractions of the form

$$\frac{1}{n}=\frac{1}{a}+\frac{1}{b} \tag{1}$$

and

$$\frac{1}{n}=\frac{1}{c}-\frac{1}{d}, \tag{2}$$

where a, b, c, d are positive integers with $a \leq b$ and $c < d$. Furthermore, we determine how many decompositions of $1/n$ of the forms (1) and (2) exist, for an arbitrary positive integer n. A fraction of the form $1/n$, where n is a nonzero integer, is called the *reciprocal* of n (see e.g. [2]).

Reciprocals and their sums remain of interest to mathematicians. In recent years, several interesting papers have been published on these topics – e.g. [3], [4], [5], [6], [7], [8], [9] and [10], [11], [12], [13], [14].

Note that an *Egyptian fraction* is a sum of finitely many rational numbers, each of which can be expressed in the form $1/q$, where q is a positive integer. For example, the Egyptian fraction $7/9$ can be written as

$$\frac{7}{9}=\frac{1}{2}+\frac{1}{4}+\frac{1}{36}.$$

Every positive rational number can be expressed as an Egyptian fraction. Such representations were already known in ancient Egypt. The first published systematic method for constructing them was described in 1202 in the Liber Abaci of Leonardo of Pisa (Fibonacci) (see [15]).

This method is called the *greedy algorithm* because at each step it selects the largest possible unit fraction that can be used in representing the remainder. For example, the Egyptian fraction representation of $7/9$ can be found by this process: The greatest unit fraction less than $7/9$ is $1/2$, the remainder is $5/18$. The greatest unit fraction less than $5/18$ is $1/4$, the remainder is $1/36$, so we get the representation stated above.

Although Egyptian fractions are no longer used in most practical applications of mathematics, modern number theorists continue to study many problems related to them. One famous unproven statement in number theory is the *Erdős–Straus conjecture* from 1948. The conjecture is that, for every integer $n \geq 2$ there exist positive integers x, y, and z for which

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z},$$

so the number $4/n$ can be written as a sum of three positive unit fractions. This conjencture has been verified by computer up to $n \leq 10^{17}$.

Basic information about Egyptian fractions can be found, for example, on Wikipedia [16]. Several papers deal with this topic (e.g. [17] and [18]), and very detailed material can be found on the website [19].

2 Four particular cases

2.1 Additive case

Let us analyze the first four cases of equation (1), i.e. for the values $n = 1, 2, 3, 4$.

► Case $n = 1$:

It follows from (1) that

$$1 = \frac{1}{a} + \frac{1}{b} \quad \longrightarrow \quad \frac{1}{b} = \frac{a-1}{a} \quad \longrightarrow \quad b = \frac{a}{a-1}.$$

Since b must be a positive integer, we require $a - 1 = 1$, which gives $a = 2$ and $b = 2$. Therefore, there is only one decomposition, namely

$$\frac{1}{1} = \frac{1}{2} + \frac{1}{2}.$$

► Case $n = 2$:

It is evident that for every $n \geq 2$ there are two decompositions of the forms

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{2n} \quad \text{and} \quad \frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}. \quad (3)$$

As a specific case, for $n = 11$ we obtain the decompositions

$$\frac{1}{11} = \frac{1}{22} + \frac{1}{22} \qquad \text{and} \qquad \frac{1}{11} = \frac{1}{12} + \frac{1}{132},$$

as mentioned in the Introduction. The decompositions (3) are identical for $n = 1$, giving $1/1 = 1/2 + 1/2$, as shown above. Hence for $n = 2$ we have two decompositions

$$\frac{1}{2} = \frac{1}{4} + \frac{1}{4} \qquad \text{and} \qquad \frac{1}{2} = \frac{1}{3} + \frac{1}{6}.$$

If there were another decomposition for $n = 2$, it would necessarily be of the form

$$\frac{1}{2} = \frac{1}{2+k} + \frac{1}{2+\ell},$$

where k, ℓ are positive integers with $k < \ell$. In this case we obtain the equation

$$(2+k)(2+\ell) = 2(2+\ell) + 2(2+k),$$

thus $4 + 2k + 2\ell + k\ell = 4 + 2\ell + 4 + 2k$, whence $k\ell = 4 = n^2$. This equation has only one solution, $k = 1$ and $\ell = 4$, which gives the decomposition $1/2 = 1/3 + 1/6$. Thus, for $n = 2$, there are only two decompositions.

► Case $n = 3$:

There exist two decompositions of the forms (3), namely

$$\frac{1}{3} = \frac{1}{6} + \frac{1}{6} \qquad \text{and} \qquad \frac{1}{3} = \frac{1}{4} + \frac{1}{12}$$

again.

If there were another decomposition for $n = 3$, it would be of the form

$$\frac{1}{3} = \frac{1}{3+k} + \frac{1}{3+\ell},$$

where k, ℓ are suitable positive integers with $k < \ell$. In this case, we obtain the equation

$$(3+k)(3+\ell) = 3(3+\ell) + 3(3+k),$$

which simplifies to $9+3k+3\ell+k\ell = 9+3\ell+9+3k$, whence $k\ell = 9 = n^2$. This equation has only one solution, $k = 1$ and $\ell = 9$, which gives the decomposition $1/3 = 1/4 + 1/12$. Hence, for $n = 3$ there are only two decompositions.

► Case $n = 4$:

There exist two decompositions of the forms (3), namely

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{8} \qquad \text{and} \qquad \frac{1}{4} = \frac{1}{5} + \frac{1}{20}.$$

If there were another decomposition for $n = 4$, it would be of the form

$$\frac{1}{4} = \frac{1}{4+k} + \frac{1}{4+\ell},$$

where k, ℓ are suitable positive integers with $k < \ell$. In this case, we obtain the equation

$$(4+k)(4+\ell) = 4(4+\ell) + 4(4+k),$$

which simplifies to $16 + 4k + 4\ell + k\ell = 16 + 4\ell + 16 + 4k$, whence $k\ell = 16 = n^2$. This equation has two solutions, $k = 1, \ell = 16$ and $k = 2, \ell = 8$, which give the decomposition of the form $1/4 = 1/5+1/20$ and another decomposition $1/4 = 1/6 + 1/12$.

Hence, for $n = 4$ there are three decompositions:

$$\frac{1}{4} = \frac{1}{8} + \frac{1}{8}, \qquad \frac{1}{4} = \frac{1}{5} + \frac{1}{20}, \qquad \frac{1}{4} = \frac{1}{6} + \frac{1}{12}.$$

2.2 Difference case

Now, let us analyze the first four cases of the equation (2), namely for $n = 1, 2, 3, 4$.

► Case $n = 1$:

From (2) we obtain

$$1 = \frac{1}{c} - \frac{1}{d} \quad \longrightarrow \quad \frac{1}{d} = \frac{1-c}{c} \quad \longrightarrow \quad d = \frac{c}{1-c}.$$

Since $c < d$ are positive integers, we must have $1 - c = 1$, hence $c = 0$, and therefore there is no such decomposition.

► Case $n = 2$:

For arbitrary $n \geq 2$, there clearly exists one decompositions of the form

$$\frac{1}{n} = \frac{1}{n-1} - \frac{1}{n(n-1)}. \tag{4}$$

For $n = 2$, there clearly exists only one decomposition of the form

$$\frac{1}{2} = \frac{1}{1} - \frac{1}{2}.$$

► Case $n = 3$:

From (2) we obtain

$$\frac{1}{3} = \frac{1}{c} - \frac{1}{d} \quad \longrightarrow \quad \frac{1}{d} = \frac{3-c}{3c} \quad \longrightarrow \quad d = \frac{3c}{3-c}.$$

Since $c < d$ are positive integers, we must have $3 - c = 1$, hence $c = 2$ and $d = 6$. Therefore, there exists only one decomposition:

$$\frac{1}{3} = \frac{1}{2} - \frac{1}{6}.$$

► Case $n = 4$:

Analogously, we get

$$\frac{1}{4} = \frac{1}{c} - \frac{1}{d} \quad \longrightarrow \quad \frac{1}{d} = \frac{4-c}{4c} \quad \longrightarrow \quad d = \frac{4c}{4-c}.$$

Since $c < d$ are positive integers, we must have $4 - c = 1$, hence $c = 3$ and $d = 12$. However, the Diophantine equation

$$4c = d(4-c)$$

does not have only this one solution. To solve it systematically, we apply the so-called *Simon's Favorite Factoring Trick* (SFFT), which consists of adding a constant to both sides of an equation and rewriting one side in product form (see e.g. [20] or [21]). Rearranging, we obtain

$$c = \frac{4d}{d+4}.$$

For c to be an integer, $d + 4$ must divide $4d$. Since (by SFFT)

$$4d = 4(d+4) - 16,$$

we deduce that $(d + 4) \mid 16$. The possible values for $d + 4$ are $1, 2, 4, 8, 16$. Substituting yields $d + 4 = 8$, so $d = 4$, $c = 16/8 = 2$, and $d + 4 = 16$, so $d = 12$ and $c = 48/16 = 3$. Therefore, there are exactly two decompositions:

$$\frac{1}{4} = \frac{1}{3} - \frac{1}{12} \quad \text{and} \quad \frac{1}{4} = \frac{1}{2} - \frac{1}{4}.$$

3 General solution

Let us analyze the two general cases again – the additive case and the difference case.

Let us derive the number of decompositions of the form (1), that is, in the form of a sum

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b},$$

where $a \leq b$. Multiplying both sides of the equality by nab, we obtain

$$ab = n(a+b) \;\longrightarrow\; ab - na - nb = 0.$$

Adding n^2 to both sides and subsequently factoring, i.e. using SFFT, we obtain

$$n^2 = (a-n)(b-n).$$

Let us define

$$x = a - n > 0, \qquad y = b - n > 0. \tag{5}$$

Then

$$xy = n^2.$$

Thus, for every positive divisor x of n^2, we have

$$a = x + n, \qquad b = y + n = \frac{n^2}{x} + n.$$

This produces all solutions (a, b) of the equation (1) in positive integers, i.e. all additive decompositions.

If the square n^2 of a positive integer n has prime factorization

$$n^2 = \prod_{i=1}^{k} {p_i}^{q_i},$$

then any positive divisor of n^2 has the form (see e.g. [22])

$${p_1}^{m_1}\, {p_2}^{m_2} \cdots {p_k}^{m_k},$$

where for integers m_i it holds

$$0 \leq m_i \leq q_i.$$

Therefore, the number $d(n^2)$ of positive divisors of n^2 (also denoted $\sigma_0(n^2)$ or $\tau(n^2)$), i.e. the so-called *divisor function*, (see e.g. [23]) is

$$d(n^2) = \prod_{i=1}^{k} (q_i + 1) . \tag{6}$$

Each positive divisor $x \mid n^2$ gives exactly one ordered pair (a, b), hence the number of ordered solutions (a, b) of the equation (1) is $d(n^2)$.

Divisor pairs (x, y) and (y, x) yield the same unordered pair $\{a, b\}$ The single symmetric case $x = y = n$ is counted only once. Thus the number $\Sigma(n)$ of unordered solutions $\{a, b\}$ of equation (1), i.e. the number of decompositions of the fraction $1/n$ into the sum $1/a + 1/b$, is

$$\Sigma(n) = \frac{d(n^2) + 1}{2} . \tag{7}$$

Similarly, for the difference case (2), that is, in the form

$$\frac{1}{n} = \frac{1}{c} - \frac{1}{d} ,$$

where $c < d$, we obtain

$$cd = n(d - c) \longrightarrow cd - nd + nc = 0 .$$

Adding and subtracting n^2 and factoring, i.e. using SFFT, we obtain

$$n^2 = (n - c)(n + d) .$$

Let us define

$$x = n - c > 0 , \qquad y = n + d > 0 . \tag{8}$$

Then $xy = n^2$, so

$$c = n - x , \qquad d = y - n = \frac{n^2}{x} - n .$$

From $c > 0$ we obtain $x < n$, so the valid choices are precisely those divisors $0 < x < n$ of n^2. Since divisors of n^2 come in symmetric pairs (x, y) with $xy = n^2$ and $x < y$, the number of such x, i.e. the number $\Delta(n)$ of ordered solutions (c, d) of equation (2), equivalently, the number of decompositions of the fraction $1/n$ into the difference $1/c - 1/d$, is

$$\Delta(n) = \frac{d(n^2) - 1}{2} . \tag{9}$$

Equivalently, both cases can be written in a unified form

$$N_{\pm}(n) = \frac{d(n^2) \pm 1}{2} , \tag{10}$$

where $N_+(n) = \Sigma(n)$ corresponds to the additive case and $N_-(n) = \Delta(n)$ matches with the difference case.

The different signs in (10) arise naturally from the symmetry of divisor pairs of n^2: in the additive case, the symmetric divisor $x = y = n$ contributes one extra solution, hence the $+1$; in the difference case, this symmetric divisor is excluded by the condition $x < n$, resulting in the -1.

4 Number of decompositions

Let us note that the numbers of divisors of a positive integer n form the sequence $\big(d(n)\big)_{n \in \mathbb{N}}$,

$$1, 2, 2, 3, 2, 4, 2, 4, 3, 4, 2, 6, 2, 4, 4, 5, 2, 6, \ldots . \tag{11}$$

This sequence is fully described as integer sequence A000005 in The On-Line Encyclopedia of Integer Sequences (see [24]).

Accordingly, the number of divisors of n that are less than n form the sequence $\big(d(n) - 1\big)_{n \in \mathbb{N}}$

$$0, 1, 1, 2, 1, 3, 1, 3, 2, 3, 1, 5, 1, 3, 3, 4, 1, 5, \ldots . \tag{12}$$

This sequence is also listed as integer sequence A032741 in The On-Line Encyclopedia of Integer Sequences (see [25]).

A *prime number* p is a positive integer greater than 1 that has exactly two distinct positive divisors, namely 1 and itself. For its square p^2, there exist only two factorizations into positive integers $q \cdot r$ with $q \leq r$:

$$p^2 = 1 \cdot p^2 \qquad \text{and} \qquad p^2 = p \cdot p ,$$

and only one factorization with $q < r$:

$$p^2 = 1 \cdot p^2 .$$

Therefore, there are only two possibilities for decomposing the fraction $1/n$ into a sum, as in (3),

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{2n} \qquad \text{and} \qquad \frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)} .$$

and only one possibility for decomposing the fraction $1/n$ into a difference, as in (4),

$$\frac{1}{n} = \frac{1}{n-1} - \frac{1}{n(n-1)} .$$

Thus, the sum decompositions of the reciprocals of prime numbers correspond to the terms equal to 2 in sequence (11), while the difference decompositions of the the reciprocals of prime numbers correspond to the terms equal to 1 in sequence (12).

For a *composite number* n, there are more than two and one possibilities for sum decompositions and more than one possibility for difference decompositions of $1/n$. Their number is derived from the sequence $\big(d(n^2)\big)_{n\in\mathbb{N}}$ and is given by (7) and (9). Let us note that the sequence $\big(d(n^2)\big)_{n\in\mathbb{N}}$ is fully described as integer sequence A048691, consisting only of odd numbers,

$$1, 3, 3, 5, 3, 9, 3, 7, 5, 9, 3, 15, 3, 9, 9, 9, 3, 15, 3, \ldots,$$

which is also listed in The On-Line Encyclopedia of Integer Sequences ([26]).

Since the total number of decompositions of fraction $1/n$ into a sum or a difference of two reciprocals, given by (7) and (9), is

$$\Sigma(n) + \Delta(n) = \frac{d(n^2)+1}{2} + \frac{d(n^2)-1}{2} = d(n^2),$$

the terms of the sequence $\big(d(n^2)\big)_{n\in\mathbb{N}}$ correspond exactly to the numbers of decompositions of $1/n$ into either a sum or a difference of two reciprocals.

The first 20 terms of these three sequences are listed in the following table:

n	$\Sigma(n)$	$\Delta(n)$	$d(n^2)$	n	$\Sigma(n)$	$\Delta(n)$	$d(n^2)$
1	1	0	1	11	2	1	3
2	2	1	3	12	8	7	15
3	2	1	3	13	2	1	3
4	3	2	5	14	5	4	9
5	2	3	3	15	5	4	9
6	5	4	9	16	5	4	9
7	2	1	3	17	2	1	3
8	4	3	7	18	8	7	15
9	3	2	5	19	2	1	3
10	5	4	9	20	8	7	15

Table 1: First twenty terms of the sequences $\Sigma(n)$, $\Delta(n)$ and $d(n^2) = \Sigma(n) + \Delta(n)$
[source: own table]

In summary, by relating the decompositions of $1/n$ to the divisor function of n^2, the sequences $\Sigma(n)$ and $\Delta(n)$ capture the complete picture: together they account for every possible way of expressing $1/n$ as the sum or the difference of two reciprocals.

5 Illustrative example

Consider the composite number

$$n = 45 = 3^2 \cdot 5^1$$

and determine the number of decompositions of the fraction $1/n = 1/45$ into ither the sum

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$

or the difference

$$\frac{1}{n} = \frac{1}{c} - \frac{1}{d}$$

of two reciprocals of positive integers.

The square $n^2 = 45^2 = 2025$ has the prime factorization

$$45^2 = (3^2 \cdot 5^1)^2 = 3^4 \cdot 5^2,$$

so by (6) it has

$$d(45^2) = (4+1)(2+1) = 15$$

positive divisors:

$$1,\ 3,\ 5,\ 9,\ 15,\ 25,\ 27,\ 45,\ 75,$$
$$81,\ 135,\ 225,\ 405,\ 675,\ 2025$$

(see e.g. [27]) that are listed in Table 2, where the ordered pairs (a, b) with $a \le b$ are highlighted in bold.

i	j	$3^i\,5^j$	divisor	quotient
0	0	$3^0\,5^0$	**1**	**2 025**
	1	$3^0\,5^1$	**5**	**405**
	2	$3^0\,5^2$	**25**	**81**
1	1	$3^1\,5^0$	**3**	**675**
	1	$3^1\,5^1$	**15**	**135**
	2	$3^1\,5^2$	75	27
2	0	$3^2\,5^0$	**9**	**225**
	1	$3^2\,5^1$	**45**	**45**
	2	$3^2\,5^2$	225	9
3	0	$3^3\,5^0$	**27**	**75**
	1	$3^3\,5^1$	135	15
	2	$3^3\,5^2$	675	3
4	0	$3^4\,5^0$	81	25
	1	$3^4\,5^1$	405	5
	2	$3^4\,5^2$	2 025	1

Table 2: Fifteen positive divisors and quotients of the square 45^2 of the composite number $45 = 3^2 \cdot 5^1$
[source: own table]

5.1 Additive case

Since in the positive integer factorization these divisors are grouped into pairs, except for the divisor 45, which is grouped with itself, there are, by (7) and Table 2,

$$\Sigma(45) = \frac{d(45^2)+1}{2} = \frac{15+1}{2} = 8$$

distinct factorizations xy of the number 45^2:

$$x = 1,\ y = 2\,025, \qquad x = 3,\ y = 675,$$
$$x = 5,\ y = 405, \qquad x = 9,\ y = 225,$$
$$x = 15,\ y = 135, \qquad x = 25,\ y = 81,$$
$$x = 27,\ y = 75, \qquad x = 45,\ y = 45.$$

Since by (5) $a - 45 = x$ and $b - 45 = y$, we obtain the following 8 pairs of equations:

$$a - 45 = 1, \qquad b - 45 = 2\,025,$$
$$a - 45 = 3, \qquad b - 45 = 675,$$
$$a - 45 = 5, \qquad b - 45 = 405,$$
$$a - 45 = 9, \qquad b - 45 = 225,$$
$$a - 45 = 15, \qquad b - 45 = 135,$$
$$a - 45 = 25, \qquad b - 45 = 81,$$
$$a - 45 = 27, \qquad b - 45 = 75,$$
$$a - 45 = 45, \qquad b - 45 = 45.$$

Therefore, for the additive decomposition of the composite number $n = 45$, we obtain the following 8 pairs of denominators (a, b):

$$(46, 2\,070),\ \ (48, 720),\ \ (50, 450),\ \ (54, 270),$$
$$(60, 180),\ \ (70, 126),\ \ (72, 120),\ \ (90, 90)$$

and the corresponding 8 decompositions:

$$\frac{1}{45} = \frac{1}{46} + \frac{1}{2\,070}, \qquad \frac{1}{45} = \frac{1}{48} + \frac{1}{720},$$
$$\frac{1}{45} = \frac{1}{50} + \frac{1}{450}, \qquad \frac{1}{45} = \frac{1}{54} + \frac{1}{270},$$
$$\frac{1}{45} = \frac{1}{60} + \frac{1}{180}, \qquad \frac{1}{45} = \frac{1}{70} + \frac{1}{126},$$
$$\frac{1}{45} = \frac{1}{72} + \frac{1}{120}, \qquad \frac{1}{45} = \frac{1}{90} + \frac{1}{90}.$$

5.2 Difference case

Now we analogously determine the number of decompositions of the fraction $1/n = 1/45$ as the difference of two reciprocals of positive integers. By (9) there are

$$\Delta(45) = \frac{d(45^2)-1}{2} = \frac{15-1}{2} = 7$$

distinct subtractive factorizations xy of the number 45^2, as shown in Table 2:

$$x = 1,\ y = 2\,025, \qquad x = 3,\ y = 675,$$
$$x = 5,\ y = 405, \qquad x = 9,\ y = 225,$$
$$x = 15,\ y = 135, \qquad x = 25,\ y = 81,$$
$$x = 27,\ y = 75.$$

Since by (8) $45 - c = x$ and $45 + d = y$, we obtain the following 7 pairs of equations:

$$45 - c = 1, \qquad 45 + d = 2\,025,$$
$$45 - c = 3, \qquad 45 + d = 675,$$
$$45 - c = 5, \qquad 45 + d = 405,$$
$$45 - c = 9, \qquad 45 + d = 225,$$
$$45 - c = 15, \qquad 45 + d = 135,$$
$$45 - c = 25, \qquad 45 + d = 81,$$
$$45 - c = 27, \qquad 45 + d = 75.$$

Therefore, for the subtractive decomposition of the composite number $n = 45$, we obtain the following 7 pairs of denominators (c, d):

$$(44, 2\,070),\ \ (42, 720),\ \ (40, 450),\ \ (36, 270),$$
$$(30, 180),\ \ (20, 126),\ \ (18, 120)$$

and the corresponding 7 decompositions:

$$\frac{1}{45} = \frac{1}{44} - \frac{1}{1\,980}, \qquad \frac{1}{45} = \frac{1}{42} - \frac{1}{630},$$
$$\frac{1}{45} = \frac{1}{40} - \frac{1}{360}, \qquad \frac{1}{45} = \frac{1}{36} - \frac{1}{180},$$
$$\frac{1}{45} = \frac{1}{30} - \frac{1}{90}, \qquad \frac{1}{45} = \frac{1}{20} - \frac{1}{36},$$
$$\frac{1}{45} = \frac{1}{18} - \frac{1}{30}.$$

In summary, for $n = 45$ we obtain eight additive and seven subtractive decompositions, which together illustrate how the divisor function of n^2 completely governs the number of possible representations.

6 Numerical verification

We solve the problem of determining the decompositions of a reciprocal into the sum and the difference of two reciprocals. We use the following simple procedure `asdeco`, which determines the total number of decompositions, the numbers of sum and difference decompositions, and explicitly lists all of them.

We performed the calculations specifically for the composite number $n = 45$ and also for the prime number $n = 47$.

```
> with(numtheory):
  asdeco:=proc(n)
  local k,m,s;
  s:=n*n;
  m:=(tau(s)+1)/2;
  print(2*m-1,"dec",m,"+dec",m-1,"-dec,");
  for k in sort(divisors(s)[1..m],`>`) do
      print(1/n=1/(n+k),"+",1/(n+(s/k)))
  end do;
  for k in sort(divisors(s)[1..m-1],`<`) do
      print(1/n=1/(n-k),"-",1/(n-(s/k)))
  end do;
  end proc:
> asdeco(45);
```

The output was a report on the total number of decompositions and 15 decompositions (presented here in two columns):

15, ”dec”, 8,”+ dec”, 7,”− dec

$$\frac{1}{45} = \frac{1}{90} + \frac{1}{90} \qquad \frac{1}{45} = \frac{1}{72} + \frac{1}{120}$$

$$\frac{1}{45} = \frac{1}{70} + \frac{1}{126} \qquad \frac{1}{45} = \frac{1}{60} + \frac{1}{180}$$

$$\frac{1}{45} = \frac{1}{54} + \frac{1}{270} \qquad \frac{1}{45} = \frac{1}{50} + \frac{1}{450}$$

$$\frac{1}{45} = \frac{1}{48} + \frac{1}{720} \qquad \frac{1}{45} = \frac{1}{46} + \frac{1}{2070}$$

$$\frac{1}{45} = \frac{1}{44} - \frac{1}{1980} \qquad \frac{1}{45} = \frac{1}{42} - \frac{1}{630}$$

$$\frac{1}{45} = \frac{1}{40} - \frac{1}{360} \qquad \frac{1}{45} = \frac{1}{36} - \frac{1}{180}$$

$$\frac{1}{45} = \frac{1}{30} - \frac{1}{90} \qquad \frac{1}{45} = \frac{1}{20} - \frac{1}{36}$$

$$\frac{1}{45} = \frac{1}{18} - \frac{1}{30}$$

The second calculation for the input

```
> asdeco(47);
```

gave the following results:

3, ”dec”, 2,”+ dec”, 1,”− dec

$$\frac{1}{47} = \frac{1}{48} + \frac{1}{2256} \qquad \frac{1}{47} = \frac{1}{48} + \frac{1}{2256}$$

$$\frac{1}{47} = \frac{1}{46} - \frac{1}{2162}$$

These computations confirm the theoretical formulas and illustrate the difference in decomposition patterns for a composite and a prime input.

7 Conclusion

In this paper we have investigated the decompositions of the reciprocal of a positive integer n into the sum and the difference of two reciprocals. We determined that the total number of all such decompositions, using Simon's Favorite Factoring Trick, is given by the number $d(n^2)$ of divisors of n^2.

We derived that for a prime number n there are only two sum decompositions

$$\frac{1}{n} = \frac{1}{2n} + \frac{1}{2n} \quad \text{and} \quad \frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)}$$

and that there exists only one difference decomposition

$$\frac{1}{n} = \frac{1}{n-1} - \frac{1}{n(n-1)}.$$

Furthermore, we derived that the number of sum decompositions is generally given by the formula

$$\Sigma(n) = \frac{d(n^2)+1}{2}$$

and that the number of difference decompositions is

$$\Delta(n) = \frac{d(n^2)-1}{2}.$$

The decomposition results were illustrated for the composite number $n = 45$. Finally, we verified these results using the basic programming features of the computer algebra system Maple 2025, where we explicitly determined the decompositions for the composite number $n = 45$ and the prime number $n = 47$.

In conclusion, it can be stated that the decomposition of the reciprocal of a positive integer into the sum and difference of two so-called Egyptian fractions belongs to the relatively simple, yet at the same time interesting parts of higher mathematics.

Area of Further Development

The results of this paper can be generalized and extended to similar cases of the decomposition of a reciprocal into the sum or difference of three or more Egyptian fractions.

For example, one can determine the number of decomposition of the reciprocal into one sum and two types of difference of three Egyptian fractions:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c},$$

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b} - \frac{1}{c},$$

$$\frac{1}{n} = \frac{1}{a} - \frac{1}{b} - \frac{1}{c}.$$

Conflict of Interest: The author has no conflicts of interest to declare that are relevant to the content of this article.

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